

# On the Spectral Gap of the antiferromagnetic Lipkin-Meshkov-Glick Model

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We study the spectral property of the antiferromagnetic Lipkin-Meshkov-Glick model. We show that at the supersymmetric point the spectrum of the model is gapped and its lower bound does not depend on the number of spins in the system. This implies that it is possible to generate specific entangled many-particle states in an ensemble of spins by adiabatic ground-state transitions in a scalable way.

## INTRODUCTION

Fifty years ago, Lipkin, Meshkov and Glick (LMG) proposed in a series of papers an exactly solvable model of interacting fermions in nuclear physics [1]. In recent years, due to the high controllability of trapped cold atomic and ionic systems, the model found applications in different areas of physics such as Bose–Einstein condensates [2], ion traps [3], in cavities [4]. The model has also been used to investigate quantum phase transitions [5] and their relations with bipartite entanglement [6],[7]. Despite the fact that it has been proved that the model is integrable through the Bethe ansatz method [8] and some properties of the system can be extracted, in principle, this method does not yield an essential simplification for calculating the spectral properties of large systems. Among the theoretical methods applied to the LMG model were the WKB method [10], the instanton technique [11], and a mapping onto a Schrödinger particle problem [9]. These methods are however only applicable in the asymptotic limit of large numbers of particles.

For many applications, e.g. the creation of many-particle entanglement by adiabatic ground-state transitions, it is important to establish a lower bound of the gap between the ground state and the rest of the spectrum. For the finite number of spins it has been shown [3], [7] that the antiferromagnetic supersymmetric LMG model (ground state with zero energy and doubly degenerate excited states in the energy spectrum) is gapped for an arbitrary even number of particles.

In this paper we present more detailed quantitative studies of the energy spectrum of this model. In particular, we find a lower bound for the spectral gap which does not depend on the number of particles. The latter result might sound paradoxical, because, it is commonly believed that the gap in the spectrum of spin-1/2 Hamiltonians decreases with increasing the number of particles. For *local Hamiltonians* in one spatial dimension with spin-1/2 it is an exact statement of the Lieb-Schultz-Mattis theorem [12]. In our model, however, the locality condition is violated and therefore this theorem is not applicable.

## COLLECTIVE SPIN COUPLING AND SUPERSYMMETRIC SPECTRUM

We begin considering the LMG model. Let us consider an even number  $N$  of spin-1/2 particles interacting through a nonlinear coupling of the collective spin  $J_i = \sum_{n=1}^N \sigma_n^{(i)}$ , where  $\sigma_n^{(i)}$  denotes the  $i$ th component of the single-particle spin. The interaction is assumed to be of second order in the total spin. The LMG Hamiltonian takes the following form

$$H = \xi (\chi_1^2 J_z^2 + \chi_2^2 J_y^2 + \lambda \chi_1 \chi_2 J_x). \quad (1)$$

For the interested reader about the LMG model we refer to the review article [13]. Hereafter, we consider only the maximum symmetric sector of the total Hilbert space, i.e. the total spin is equal  $N/2$ . The parameters  $\chi_{1,2}, \lambda$  are assumed to be positive constants. At the point where  $\chi_2 = 0$ , depending on the sign of the  $\xi$  the ground state is ferromagnetic or antiferromagnetic for  $\xi < 0$  and  $\xi > 0$  respectively. In the following we consider only the antiferromagnetic case. We note that by the rotation  $\exp(i\frac{\pi}{2}J_x)$  the Hamiltonian (1) can be transformed into the form  $\chi_1^2 J_y^2 + \chi_2^2 J_z^2 + \lambda \chi_1 \chi_2 J_x$  and therefore, we may assume without loss of generality that  $\chi_1 \geq \chi_2$ . As has been shown in [3] the spectrum of the antiferromagnetic LMG model becomes supersymmetric at the point  $\lambda = 1$ .

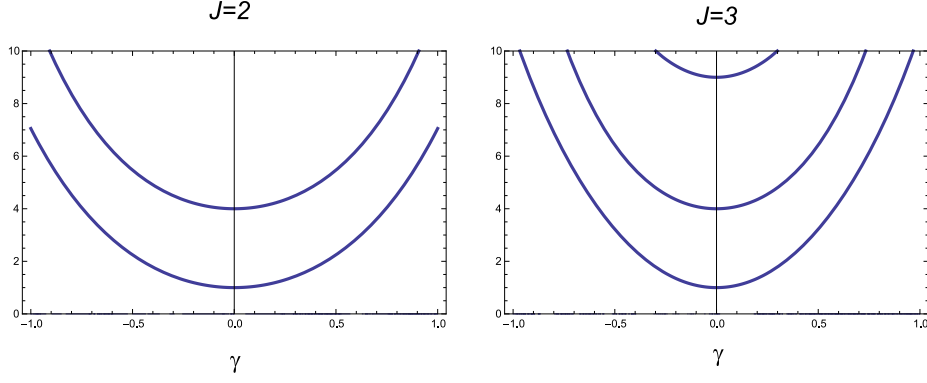


FIG. 1: (color online) Typical supersymmetric spectra of the LMG model for  $J = 2$  and  $J = 3$  depending on the parameter  $\gamma$ . The ground state zero energy not shown

The analysis of the spectrum of the Hamiltonian operator (1) can be greatly simplified by introducing new parameters,

$$\begin{aligned}\chi_1 &= \Omega_0 \cosh \gamma, \\ \chi_2 &= \Omega_0 \sinh \gamma,\end{aligned}\tag{2}$$

where

$$\Omega_0^2 = \chi_1^2 - \chi_2^2, \quad \tanh \gamma = \frac{\chi_2}{\chi_1}\tag{3}$$

and making use of the expression for a hyperbolic rotation about the  $x$ -axis with the parameter  $\gamma$

$$\exp(-\gamma J_x) J_z \exp(\gamma J_x) = J_z \cosh \gamma + i J_y \sinh \gamma.\tag{4}$$

Thanks to this transformation, the Hamiltonian can be factorized as follows

$$\begin{aligned}H &= \Omega_0^2 (J_z \cosh \gamma + i J_y \sinh \gamma) (J_z \cosh \gamma - i J_y \sinh \gamma) = \\ &= \Omega_0^2 \exp(-\gamma J_x) J_z \exp(2\gamma J_x) J_z \exp(-\gamma J_x)\end{aligned}\tag{5}$$

We set  $\Omega_0 = 1$  throughout. We see that for an arbitrary value of  $\gamma$  all eigenvalues  $E$  of the Hamiltonian (5) satisfy  $E \geq 0$ , and in addition to that for even number  $N$  of spin-1/2, i.e. integer angular momentum  $J = N/2$ , there is ground state with  $E = 0$ :

$$|\Psi_0\rangle = \frac{1}{\sqrt{P_J(\cosh 2\gamma)}} \exp(\gamma J_x) |m_z = 0\rangle\tag{6}$$

with  $P_J$  being Legendre polynomials (for more details, see [3] and [7]).

Figure 1 shows the typical spectra for  $J = 2$  and  $J = 3$  depending on anisotropy parameter  $\gamma$ . As one can see from Eq. (5) the spectrum of the Hamiltonian is symmetric with respect to the  $\gamma$ .

A glance at the spectrum of the Hamiltonian as a function of  $\gamma$  reveals (see Fig.1) that the spectrum consists of pairwise degenerate levels and a nondegenerate single state (the ground state) which signals the presence of a supersymmetry. In addition to that, we observe (Fig.2) that the gap does not close for large values  $N$ . The Hamiltonian (5) has, for  $\gamma \neq 0$ , no obvious symmetries other than the rotational symmetry around the  $x$ -axis by  $\pm\pi$  and the spin-flip symmetry. It is clear that those symmetries cannot be responsible for this kind of spectrum.

Now let us move on to an explanation of these observations. We begin by the Schwinger-boson representation of the angular momentum operators in terms of two bosonic operators  $a$  and  $b$

$$\begin{aligned}J_x &= \frac{1}{2}(ab^\dagger + ba^\dagger), \\ J_y &= \frac{1}{2i}(ba^\dagger - ab^\dagger), \\ J_z &= \frac{1}{2}(a^\dagger a - b^\dagger b).\end{aligned}\tag{7}$$

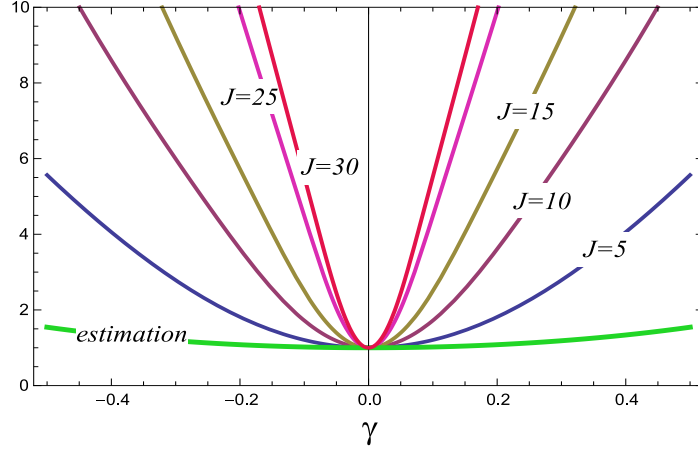


FIG. 2: (color online) Gap of the antiferromagnetic LMG model for different number of particles ( $J = 5, 10, 15, 25$  and  $30$ ) as a function of anisotropy parameter  $\gamma$ . The green line shows the analytical estimation ( $\cosh 2\gamma$ ) for the gap.

For our purposes it is convenient to use another unitarily equivalent form of the Hamiltonian (1), namely

$$H = J_x^2 \cosh^2 \gamma + J_y^2 \sinh^2 \gamma + J_z \cosh \gamma \sinh \gamma. \quad (8)$$

This Hamiltonian can be transformed into the following block diagonal form

$$H = \begin{bmatrix} \mathbf{H}_{even} & \mathbf{0} \\ \mathbf{0} & \mathbf{H}_{odd} \end{bmatrix} \quad (9)$$

by grouping together its elements with even and odd bosonic excitations. In accordance with our partition  $H_{even}$  is a matrix with the dimension  $(n+1) \times (n+1)$ . It acts on states with even number of bosons such as  $(a^\dagger)^k (b^\dagger)^{N-k} |0\rangle$  where  $k$  is an even number. And  $H_{odd}$  has  $n \times n$  dimension and acts on states with odd number of bosons such as  $(a^\dagger)^n (b^\dagger)^{N-n} |0\rangle$  where  $n$  is an odd number. The Hamiltonian (8) for  $N = 2n$  particles has dimension  $(N+1) \times (N+1)$ . The SUSY spectrum of the system must be twofold degenerate except for ground state i.e. the total number of states must be an odd number. Therefore we can see immediately that for an odd number of particles the SUSY should be broken. The decomposition (9) allows us to introduce a "fermionic" number  $F$ : if  $F = 0$  one has even excitations and  $F = 1$  odd excitations. At the SUSY point ( $\lambda = 1$ ) one can introduce the following two off-diagonal (in the basis of odd and even bosonic excitations) operators (supercharges)

$$Q_1 = \sigma_x J_x \cosh \gamma + \sigma_y J_y \sinh \gamma = \begin{bmatrix} \underbrace{\mathbf{0}}_{(n+1) \times (n+1)} & J_x \cosh \gamma - i J_y \sinh \gamma \\ J_x \cosh \gamma + i J_y \sinh \gamma & \underbrace{\mathbf{0}}_{n \times n} \end{bmatrix}, \quad (10)$$

$$Q_2 = \sigma_y J_x \cosh \gamma - \sigma_x J_y \sinh \gamma = \begin{bmatrix} \underbrace{\mathbf{0}}_{(n+1) \times (n+1)} & -J_y \sinh \gamma - i J_x \cosh \gamma \\ -J_y \sinh \gamma + i J_x \cosh \gamma & \underbrace{\mathbf{0}}_{(n+1) \times (n+1)} \end{bmatrix}. \quad (11)$$

One can check easily that the LMG Hamiltonian at the SUSY point can be represented as square of  $Q_{1,2}$  namely

$$H = Q_1^2 = Q_2^2, \quad (12)$$

and in addition to that  $Q_1$  and  $Q_2$  anticommute with each other

$$\{Q_1, Q_2\} = 0. \quad (13)$$

Combining relations (12) and (13) in the form [14]

$$\begin{aligned} \{Q_i, Q_j\} &= 2\delta_{ij}H, \quad i, j = 1, 2 \\ [Q_i, H] &= 0, \end{aligned} \quad (14)$$

we arrive at a simple superalgebra, i.e., an algebra which includes both commutation and anticommutation relations.

The above discussion can be made more concrete by looking at the case when  $N = 4$  ( $J = 2$ ). In this case, the LMG Hamiltonian (8) after the renumbering states with even and odd excitations takes the following block diagonal form

$$H = \begin{bmatrix} e^{-2\gamma} & \sqrt{\frac{3}{2}} & 0 & 0 & 0 \\ \sqrt{\frac{3}{2}} & 3 \cosh 2\gamma & \sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}} & e^{2\gamma} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} (5 \cosh 2\gamma + \sinh 2\gamma) & \frac{3}{2} \\ 0 & 0 & 0 & \frac{3}{2} & \frac{1}{2} (5 \cosh 2\gamma - \sinh 2\gamma) \end{bmatrix}.$$

The corresponding supercharges are

$$Q_1 = \begin{bmatrix} 0 & 0 & 0 & e^{-\gamma} & 0 \\ 0 & 0 & 0 & \sqrt{\frac{3}{2}} e^{\gamma} & \sqrt{\frac{3}{2}} e^{-\gamma} \\ 0 & 0 & 0 & 0 & e^{\gamma} \\ e^{-\gamma} & \sqrt{\frac{3}{2}} e^{\gamma} & 0 & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}} e^{-\gamma} & e^{\gamma} & 0 & 0 \end{bmatrix},$$

and

$$Q_2 = i \begin{bmatrix} 0 & 0 & 0 & -e^{-\gamma} & 0 \\ 0 & 0 & 0 & -\sqrt{\frac{3}{2}} e^{\gamma} & -\sqrt{\frac{3}{2}} e^{-\gamma} \\ 0 & 0 & 0 & 0 & -e^{\gamma} \\ e^{-\gamma} & \sqrt{\frac{3}{2}} e^{\gamma} & 0 & 0 & 0 \\ 0 & \sqrt{\frac{3}{2}} e^{-\gamma} & e^{\gamma} & 0 & 0 \end{bmatrix}.$$

A simple matrix calculation shows that these matrices satisfy the SUSY algebra (14).

Because the Hamiltonian is  $Q_1^2 = Q_2^2$  its eigenvalues cannot be negative. Any states with zero energies are therefore ground states, and must be annihilated by both  $Q_1$  and  $Q_2$ . For the excited states, however, in general, no further information can be gotten from the SUSY algebra (14), besides that they are two-fold degenerate [14].

In the following we show that by using the representation (5) the degeneracy and an estimation for the gap can be obtained through direct calculations.

### SPECTRAL GAP OF THE LMG HAMILTONIAN

It is easy to see from the Eq. (5) that the LMG Hamiltonian is similar to the non-hermitian Hamiltonian

$$\begin{aligned} H_n &= \exp(-\gamma J_x) H \exp(\gamma J_x) = \exp(-2\gamma J_x) J_z \exp(2\gamma J_x) J_z = \\ &= J_z^2 \cosh 2\gamma + i J_y J_z \sinh 2\gamma, \end{aligned} \quad (15)$$

From this form, one can see clearly that this Hamiltonian has a null state  $|m_z = 0\rangle$  ( $J_z |m_z = 0\rangle = 0$  for an integer angular momentum). The second term gives rise to a coupling between  $|m_z\rangle$  and  $|m_z \pm 1\rangle$  states. The operator  $H_n$  can be represented as a block matrix in a basis where  $J_z$  is diagonal,

$$H_n = \begin{pmatrix} H_- & 0 & 0 \\ \langle a| & 0 & \langle a| \\ 0 & 0 & H_+ \end{pmatrix}, \quad (16)$$

where  $H_+$  and  $H_-$  are permutation equivalent real  $J \times J$  matrices, i.e. they have the same spectrum,  $\langle a|$  is the transposed vector which connects subspace with negative and positive values of the magnetic quantum numbers. For

our purposes, the generic form of  $\langle a|$  is not relevant. For example, the Hamiltonian  $H_n$ , for  $J = 2$ , takes the following form

$$H_n = \begin{pmatrix} 4 \cosh 2\gamma & \sinh 2\gamma & 0 & 0 & 0 \\ -2 \sinh 2\gamma & \cosh 2\gamma & 0 & 0 & 0 \\ 0 & -\frac{\sqrt{6}}{2} \sinh 2\gamma & 0 & -\frac{\sqrt{6}}{2} \sinh 2\gamma & 0 \\ 0 & 0 & 0 & \cosh 2\gamma & -2 \sinh 2\gamma \\ 0 & 0 & 0 & \sinh 2\gamma & 4 \cosh 2\gamma \end{pmatrix},$$

where

$$H_- = \begin{pmatrix} 4 \cosh 2\gamma & \sinh 2\gamma \\ -2 \sinh 2\gamma & \cosh 2\gamma \end{pmatrix}, \quad H_+ = \begin{pmatrix} \cosh 2\gamma & -2 \sinh 2\gamma \\ \sinh 2\gamma & 4 \cosh 2\gamma \end{pmatrix}$$

and

$$\langle a| = \left( -\frac{\sqrt{6}}{2} \sinh 2\gamma, 0 \right).$$

One can see that  $H_+$  and  $H_-$  are permutation equivalent, i.e.

$$H_- = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} H_+ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The matrix elements of  $H_-$  with respect to the basis  $J_z$  are

$$(H_-)_{m,m'} = m^2 \delta_{mm'} \cosh(2\gamma) + \frac{m'}{2} \sinh 2\gamma \left[ \delta_{m,m'+1} \sqrt{(J-m')(J+m'+1)} - \delta_{m,m'-1} \sqrt{(J+m')(J-m'+1)} \right] \quad (17)$$

$$m, m' = -J, -J+1, \dots, -1.$$

The spectrum of the  $H_n$  can be obtained from the following algebraic equation

$$\lambda \det(\lambda \cdot \mathbb{1}_{J \times J} - H_+) \det(\lambda \cdot \mathbb{1}_{J \times J} - H_-) = 0. \quad (18)$$

Since  $H_+$  and  $H_-$  have the same spectrum, the spectrum of  $H_n$  is doubly degenerate except for the eigenstate  $|m_z = 0\rangle$ . In other words, we have shown that the initial Hamiltonian (5) in the case of integer angular momentum has a supersymmetric spectrum [3]. Hence, to find the excited spectrum of the LMG Hamiltonian it is enough to find the spectrum of the  $H_+$  or  $H_-$ . The ground state energy of the  $H_+$  or  $H_-$  will correspond to the gap of the original Hamiltonian.

As one can see from the Eq (17), the matrix  $H_-$  is a real tridiagonal matrix and its upper and lower sub-diagonals have opposite sign.

Let us consider a real tridiagonal matrix with real spectrum

$$A = \begin{bmatrix} \alpha_1 & -\beta_1 & & & \\ \gamma_1 & \alpha_2 & -\beta_2 & & \\ & \gamma_2 & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & -\beta_{n-1} \\ & & & & \gamma_{n-1} & \alpha_n \end{bmatrix}. \quad (19)$$

with the matrix elements

$$\beta_i, \gamma_i > 0, \quad i = 1, 2, \dots, n-1. \quad (20)$$

By the similarity transformation

$$A' = T A T^{-1}, \quad (21)$$

one can transform this matrix into a more symmetric form

$$A' = \begin{bmatrix} \alpha_1 & -\sqrt{\beta_1\gamma_1} & & & \\ \sqrt{\beta_1\gamma_1} & \alpha_2 & -\sqrt{\beta_2\gamma_2} & & \\ & \sqrt{\beta_2\gamma_2} & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots \\ & & & \sqrt{\beta_{n-1}\gamma_{n-1}} & -\sqrt{\beta_{n-1}\gamma_{n-1}} \\ & & & & \alpha_n \end{bmatrix}. \quad (22)$$

where  $T$  is a diagonal matrix with diagonal entries

$$t_1 = 1, \quad \frac{t_i}{t_{i+1}} = \sqrt{\frac{\beta_i}{\gamma_i}}. \quad (23)$$

The real matrix  $A'$  can be written as

$$A' = \frac{A' + A'^T}{2} + i \frac{A' - A'^T}{2i}, \quad (24)$$

where the real part  $\frac{A' + A'^T}{2}$  is diagonal. Let  $\lambda_n(A') = \lambda_n(A) > 0$  and  $|\varphi_n\rangle$  are smallest eigenvalue and corresponding normalized eigenvector of the  $A'$ . An estimation for  $\lambda_n$  can be obtained from the decomposition (24). Indeed,

$$\lambda_n(A) = \langle \varphi_n | A' | \varphi_n \rangle = \langle \varphi_n | \frac{A' + A'^T}{2} | \varphi_n \rangle \geq \alpha_n = \min(\alpha_k) \quad (25)$$

Applying this estimation, we obtain the desired result a lower bound for the gap

$$\Delta E = E_n(H_-) \geq \cosh 2\gamma \quad (26)$$

between the first two eigenvalues of the antiferromagnetic supersymmetric LMG model.

## DISCUSSION

We have shown that the antiferromagnetic supersymmetric LMG model with even number of spins is gapped. Surprisingly the obtained lower bound of the gap does not depend on the total number of spins in the system. This means that in contrast to one dimensional local systems with spin-1/2 where the gap decreases with increasing number of particles for spins-1/2 systems interacting through a collective coupling of the Lipkin-Meshkov-Glick type this is in general not the case. Figure 2 shows that our estimate is almost exact for small  $\gamma$ . Therefore, one can think naively that the result (26) for small  $\gamma$  is a perturbative estimate. The above consideration shows, however, that usual perturbation theory is not applicable at the supersymmetric point. We note that for small  $\gamma$  the LMG model can be considered as an example of a spin tunnelling system [10]. In order to calculate the gap of the spectrum for small  $\gamma$ , one has to find the tunnelling splitting of pairwise degenerate excited states. The supersymmetric models are special ones due to the fact that in any order of  $\gamma$  the tunnelling splitting must disappear. It is worth noting, however, that even if it is would be possible to find some estimation for the small  $\gamma$  (by some perturbative arguments), it will still be insufficient for applications such as creation of many-particle entangled states by adiabatic ground-state transitions, where an adiabatic process starts at  $\gamma \rightarrow \infty$  (ground state is a product state) and end up in an entangled state at  $\gamma = 0$ .

An interesting extension of this work will be to study the LMG model with broken supersymmetry ( $\lambda \neq 1$ ) and the dependence of the adiabatic condition on the size of the system. A detailed study of this problem will be given elsewhere.

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